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Spinning gas clouds without vorticity: the two missing integrals

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Abstract

We consider an ordinary differential reduction of the gas-dynamical equations proposed by Ovsiannikov and Dyson, representing a tri-axial ellipsoidal gas cloud rotating as it expands into the vacuum. For a monatomic gas ($\gamma = \frac{5}{3}$) without vorticity, the system has the Painlevé property and is integrable, at least in cases of rotation around a fixed axis. We present preliminary results concerning fully general states of rotation.

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1. Introduction

This paper deals with the ordinary differential reduction of the equations of gas dynamics, first considered by Ovsiannikov (1956) and by Dyson (1968), under the additional restricting assumptions of an ideal monatomic gas (with adiabatic index $\gamma = \frac{5}{3}$) flowing without vorticity. The governing equations then assume the form:

$$\operatorname{div}\vec{v} = \frac{-1}{(\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\ln T_e \tag{1.1a}$$

$$\partial_t\vec{v} = T_e\vec{\nabla}S - \vec{\nabla}\left(\frac{\vec{v}^2}{2} + \frac{\gamma T_e}{\gamma-1}\right) \tag{1.1b}$$

$$\partial_t S + \vec{v}\cdot\vec{\nabla}S = 0 \tag{1.1c}$$

which are the laws of conservation of mass, momentum and entropy. T_e is the temperature, normalized in such a way that the specific enthalpy $H = \gamma T_e/(\gamma-1)$, $\vec{v} = \mathrm{d}\vec{x}/\mathrm{d}t$ the velocity, S the entropy and $\mathrm{d}/\mathrm{d}t$ represents the comoving derivative $\partial_t + \vec{v}\cdot\vec{\nabla}$.

Ovsiannikov and Dyson remarked that this system admits an ordinary differential equation (ODE) reduction where S is a quadratic function of Cartesian coordinates $\vec{x}$, $\vec{v}$ is a linear function of $\vec{x}$ (with coefficients which are a function of time t) and T_e is a function of time only. Since the density ρ is given by

$$\ln\rho = \frac{1}{(\gamma-1)}\ln T_e - S \tag{1.2}$$

for an ideal gas, this ODE reduction describes an ellipsoidal gas cloud with a Gaussian density profile, expanding and rotating, with a linear velocity distribution:

$$\vec{v} = V[\vec{x}]. \tag{1.3}$$

As the vorticity is assumed to vanish, the 3×3 matrix V is symmetric.

The ODE reduction may also be interpreted as representing Hamiltonian motion of a point mass in nine-dimensional Euclidean space (Dyson 1968), in the potential $T_e/(\gamma - 1)$, the specific internal energy of the fluid.

In a recent paper (Gaffet 2000) I have considered the restricted case where V is block-diagonal, in which the gas cloud is rotating around a principal axis of fixed direction, and I have shown Liouville integrability of the equivalent Hamiltonian motion; furthermore, the corresponding general solution was shown to admit expansions of Kowalevski–Painlevé type (Kowalevski 1889, 1890, Painlevé 1902, Ince 1956) in terms of an independent variable u which is distinct from time:

$$u = \int T_e(t)\,\mathrm{d}t. \tag{1.4}$$

In this paper I will try to generalize these results to fully arbitrary states of rotation, where the matrix V is no longer block-diagonal. In particular, I will give a generalization of the new integral of the motion I_6 obtained in my earlier work, and present a candidate for the last missing integral.

2. The model

The model was originally described by Ovsiannikov and by Dyson in the Lagrangian formalism, where the independent variables are the Lagrangian coordinates $\vec{\alpha}$, characterized by the property

$$\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t} = 0. \tag{2.1}$$

In a Eulerian formalism the Lagrangian variables may be retrieved in the following way: a matrix $F(t)$ is introduced, which satisfies the equation

$$\dot{F} = VF \tag{2.2}$$

(where a dot symbolizes $\mathrm{d}/\mathrm{d}t$). We have

$$\ln(\det F) = \int \mathrm{Tr}(V)\,\mathrm{d}t \tag{2.3}$$

so that, starting from a regular initial value of F, F remains regular.

It is easy to show that the column vector

$$[\vec{\alpha}] = F^{-1}[\vec{x}] \tag{2.4}$$

constitutes a set of Lagrangian coordinates. Equation (2.4) expresses the fact that Cartesian and Lagrangian coordinates are linearly related (which was the basis of the original derivations of the model). It can be shown that the equations of gas dynamics (1.1) entail the following equation of motion for F:

$$F_T\ddot{F} = T_e \tag{2.5}$$

where the lower index T denotes transposition. As a consequence of the continuity equation (1.1a), the temperature T_e is related to the determinant of F:

$$\det F = 1/T_e^{3/2} \qquad \left(\text{using } \gamma = \tfrac{5}{3}\right). \tag{2.6}$$

It is worth noting that, whenever F solves (2.5) and (2.6), F_T constitutes another solution, that property is known as the duality principle of Dedekind (1860).

Another immediate consequence of (2.5) is the constancy of the two matrices J and K:

$$\begin{aligned} J &= F\dot{F}_T - \dot{F}F_T \\ K &= F_T\dot{F} - \dot{F}_T F \end{aligned} \tag{2.7}$$

which represent angular momentum and vorticity, respectively. (In this paper, K will be taken to be zero.)

It was also noted by Dyson that (2.5) may be rewritten in the form

$$\ddot{F}_{ij} + \frac{1}{(\gamma-1)}\frac{\partial T_e}{\partial F_{ij}} = 0 \tag{2.8}$$

(where T_e, as a function of F, is defined by equation (2.6)), which clearly represents the Hamiltonian motion in the nine-dimensional Euclidean space of coordinates F_{ij}, in the potential $T_e/(\gamma-1)$. There follows the law of conservation of energy:

$$E = \tfrac{1}{2}\,\mathrm{Tr}(\dot{F}\dot{F}_T) + \frac{T_e}{(\gamma-1)}. \tag{2.9}$$

3. The symmetry group: $O(3) \times O(3)$

The most general 3×3 matrix F may be decomposed in the form

$$F = O_1 D O_2 \tag{3.1}$$

where D is diagonal ($D = (D_1, D_2, D_3)$), and O_1, O_2 are two orthogonal matrices, operating in the spaces of Cartesian and Lagrangian coordinates, respectively. The diagonal part may be found, for example, through diagonalization of the symmetric matrix FF_T:

$$FF_T = O_1 D^2 O_{1T}. \tag{3.2}$$

Clearly, together with F, $\Omega_1 F \Omega_2$ also solves the equations of motion, Ω_1 and Ω_2 being arbitrary (constant) orthogonal matrices: the system presents two independent symmetry groups $O(3)$, whose product is isomorphic to $O(4)$.

Equation (3.1) may be viewed as a transformation to new coordinates D_1, D_2, D_3, supplemented by the six coordinates of $O(3) \times O(3)$. The latter will not appear explicitly in the equations, provided that we use the angular velocities 'in the moving frame', defined by

$$\begin{aligned} \dot{O}_1 &= -O_1\omega \\ \dot{O}_2 &= \varphi O_2. \end{aligned} \tag{3.3}$$

Here we use the convention that ω, φ, etc are the antisymmetric matrices dual to the 3-vectors $\vec{\omega}$, $\vec{\varphi}$, etc

$$\omega_{ij} = \tfrac{1}{2}\varepsilon_{ijk}\omega_k. \tag{3.4}$$

The representations of $\dot{F}$ and $\ddot{F}$ in the moving frame read

$$\dot{F} = O_1 H O_2 \tag{3.5a}$$

$$\ddot{F} = O_1 T_e D^{-1} O_2 \tag{3.5b}$$

and the equation of motion (2.5) thus becomes the system

$$\dot{D} + (D\varphi - \omega D) = H \tag{3.6a}$$

$$\dot{H} + (H\varphi - \omega H) = T_e D^{-1} \tag{3.6b}$$

which is equivalent to Dyson's equation (35).

The angular momenta j, k in the moving frame:

$$\begin{aligned} j &= O_{1T} J O_1 \\ k &= O_2 K O_{2T} \end{aligned} \tag{3.7}$$

are given explicitly by

$$\begin{aligned} j &= D^2\omega + \omega D^2 - 2D\varphi D \\ k &= D^2\varphi + \varphi D^2 - 2D\omega D. \end{aligned} \tag{3.8}$$

Their duals, $\vec{j}$ and $\vec{k}$, satisfy equations of motion which express conservation of J and K:

$$\begin{aligned} \frac{\mathrm{d}\vec{j}}{\mathrm{d}t} &= \vec{j} \wedge \vec{\omega} \\ \frac{\mathrm{d}\vec{k}}{\mathrm{d}t} &= \vec{k} \wedge \vec{\varphi} \end{aligned} \tag{3.9}$$

that constitutes the off-diagonal part of the equation of motion (3.6).

The kinetic energy, in the moving frame, is expressed by

$$\begin{aligned} \vec{v}^2 &= \mathrm{Tr}(\dot{F}\dot{F}_T) = \mathrm{Tr}(\dot{D}^2 + 2\omega D\varphi D - (\omega^2 + \varphi^2)D^2) \\ &= \dot{\vec{D}}^2 + \vec{j}\cdot\vec{\omega} + \vec{k}\cdot\vec{\varphi} \end{aligned} \tag{3.10}$$

(where $\vec{D}$ is the vector with components D_1, D_2, D_3). If, following Dyson, one rewrites the last two terms in a manifestly $O(4)$ invariant way, the analogy with a four-dimensional top becomes apparent.

The relations (3.8) between angular momenta and angular velocities, read explicitly

$$\begin{aligned} j_1 &= \omega_1(D_2^2 + D_3^2) - 2\varphi_1 D_2 D_3 \\ k_1 &= -2\omega_1 D_2 D_3 + \varphi_1(D_2^2 + D_3^2) \end{aligned} \tag{3.11}$$

together with the equations deducible by circular permutation; and, when there is no vorticity, they simplify to

$$\frac{j_1}{\omega_1} = \frac{(D_2^2 - D_3^2)^2}{(D_2^2 + D_3^2)}. \tag{3.12}$$

Finally, the diagonal part of equation (3.6), which is the only part where the dynamical effect of pressure manifests itself, reads

$$\ddot{D}_1 + [(\omega_1^2 + \varphi_1^2) - (\vec{\omega}^2 + \vec{\varphi}^2)]D_1 + 2\omega_3\varphi_3 D_2 + 2\omega_2\varphi_2 D_3 = \frac{T_e}{D_1} \tag{3.13}$$

(and the equations deducible by permutation).

4. The symmetry group: (T^*)

It has been shown by Ovsiannikov (1982) and by Gaffet (1981, 1983, 1996) that the most general flow of a monatomic gas (of adiabatic index $\gamma = \frac{5}{3}$) presents a discrete symmetry:

$$\begin{aligned} t^* &= -1/t \\ \vec{x}^* &= \vec{x}/t \\ \vec{v}^* &= (\vec{v}t - \vec{x}) \\ T_e^* &= t^2 T_e. \end{aligned} \tag{4.1}$$

This gives rise to an $SL(2)$ group of Möbius transformations of time, denoted by (T^*), with generators G_1, G_2, G_3:

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t} \\ G_2 &= 2t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i} - v_i\frac{\partial}{\partial v_i} - 2T_e\frac{\partial}{\partial T_e} \\ G_3 &= t^2\frac{\partial}{\partial t} + tx_i\frac{\partial}{\partial x_i} - v_i^*\frac{\partial}{\partial v_i} - 2tT_e\frac{\partial}{\partial T_e}. \end{aligned} \tag{4.2}$$

An immediate consequence of the symmetry is that the polar moment of inertia of an isolated mass of monatomic gas, must be a quadratic function of time. In the present problem, the moment of inertia is just $R^2 = \mathrm{Tr}(FF_T) = \mathrm{Tr}(D^2)$, so that one can write

$$R^2/2 = Et^2 + \Sigma t + E^* \tag{4.3}$$

where E (the energy), Σ and E^* are three constants; this result was first derived by Anisimov and Lysikov (1970). Thus the radial part $(R(t))$ of the equivalent Hamiltonian motion is known independently, and ought to be removed. This is achieved through the reduction of the nine-dimensional Euclidean motion to a new Hamiltonian motion on the eight-dimensional unit sphere $R^2 = 1$, by means of the transformation, which may be viewed as a generalized version of equation (4.1):

$$\begin{aligned} t_s &= \int \frac{\mathrm{d}t}{R^2} \\ \vec{x}_s &= \frac{\vec{x}}{R} \\ T_{es} &= R^2 T_e. \end{aligned} \tag{4.4}$$

The result is Hamiltonian motion on the unit sphere, in a potential $T_{es}/(\gamma - 1)$, and it is (T^*) invariant. The new value $\hat{E}$ of the energy is the (T^*) invariant that may be constructed from E, E^* and Σ, and is the discriminant of equation (4.3):

$$2\hat{E} = 4EE^* - \Sigma^2. \tag{4.5}$$

This value of $\hat{E}$ may be physically interpreted as resulting from the removal from E of the kinetic energy $\dot{R}^2/2$ of radial motion, followed by a rescaling of E by a factor $\mathrm{d}t/\mathrm{d}t_s = R^2$, in view of the fact that the function $R(t)$ satisfies the differential equation:

$$\frac{\hat{E}}{R^2} = E - \frac{\dot{R}^2}{2}. \tag{4.6}$$

Let us now rewrite the complete set of equations in a (T^*) invariant way; for reasons that will become apparent later, we shall use the Eulerian formalism, based on the matrix V, rather than the Lagrangian one (based on F).

The essential result is that a (T^*) invariant matrix (denoted by $\tilde{V}_s$) may be derived from V by first taking its traceless part $\tilde{V}$ (called the 'deformation tensor' in fluid dynamics), then rescaling it by a factor T_e:

$$\begin{aligned} V &= \tilde{V} + S/3 \qquad (S = \mathrm{Tr}(V)) \\ \tilde{V}_s &= \tilde{V}/T_e. \end{aligned} \tag{4.7}$$

The equation of motion for V:

$$\dot{V} + V^2 = O_1 T_e D^{-2} O_{1T} \tag{4.8}$$

gives rise to the following one for $\tilde{V}_s$:

$$\frac{\mathrm{d}\tilde{V}_s}{\mathrm{d}u} + \tilde{V}_s^2 - \tfrac{1}{3}\,\mathrm{Tr}\big(\tilde{V}_s^2\big) = O_1 D_s^{-2} O_{1T} - \tfrac{1}{3}\,\mathrm{Tr}\big(D_s^{-2}\big) \tag{4.9}$$

where

$$D_s = D\sqrt{T_e}. \tag{4.10}$$

Transforming to the moving frame by means of

$$\tilde{V}_s = O_1 v_s O_{1T} \tag{4.11}$$

the equation of motion becomes

$$\frac{\mathrm{d}v_s}{\mathrm{d}u} + v_s^2 - \tfrac{1}{3}\,\mathrm{Tr}\big(v_s^2\big) = [\omega_s, v_s] + D_s^{-2} - \tfrac{1}{3}\,\mathrm{Tr}\big(D_s^{-2}\big) \tag{4.12}$$

where

$$\omega_s = \frac{\omega}{T_e} \tag{4.13}$$

is determined (see equations (3.5*a*) and (3.6*a*)) by the off-diagonal elements of v_s and by D_s,

$$\omega_{s12} = v_{s12}\frac{\big(D_{s1}^2 + D_{s2}^2\big)}{\big(D_{s1}^2 - D_{s2}^2\big)} \qquad \text{(and circular permutation).} \tag{4.14}$$

The diagonal elements of v_s determine the evolution of D_s:

$$\frac{\mathrm{d}}{\mathrm{d}u}\ln D_{s1} = v_{s11} \qquad \text{(and circular permutation).} \tag{4.15}$$

5. Some integrals of hyperspherical geodesic motion

The aim of this paper is the determination of new integrals of the motion, in view of establishing the Liouville integrability of our system. As we are dealing with Hamiltonian motion on the 8-sphere, eight commuting integrals should be needed, while six are already known:

$$\hat{E},\ \vec{J}^2,\ J_3 \quad \text{and the vanishing } K_1, K_2, K_3.$$

Thus two missing integrals remain to be found.

We start with the observation that the pressure force term (the traceless part of D_s^{-2}) in equation (4.12) is of degree zero in the velocities, or momenta, whereas the rest of the equation (which merely describes geodesic motion) is quadratic. As a result, *the terms of highest degree* (in the momenta) *in any integral of the motion, must be integrals of geodesic motion*. This is the motivation for the following study of hyperspherical free motion, which is described by either one of the equations:

$$\ddot{F} = 0 \tag{5.1}$$

$$\dot{V} + V^2 = 0 \tag{5.2}$$

$$\frac{\mathrm{d}\tilde{V}_s}{\mathrm{d}u} + \tilde{V}_s^2 - \tfrac{1}{3}\,\mathrm{Tr}\big(\tilde{V}_s^2\big) = 0 \tag{5.3}$$

$$\frac{\mathrm{d}v_s}{\mathrm{d}u} + v_s^2 - \tfrac{1}{3}\,\mathrm{Tr}\big(v_s^2\big) = [\omega_s, v_s]. \tag{5.4}$$

The integrals of free motion are, of course: $\dot{F}$ and $F_{ij}\dot{F}_{kl} - \dot{F}_{ij}F_{kl}$, i.e. the difference of tensor products $F \times \dot{F} - \dot{F} \times F$; but only special combinations of them, such as $J = (F\dot{F}_T - \dot{F}F_T)$ and K, are relevant to the present problem.

Let us start with the simplest variable, the rescaled temperature: $T_{es} = R^2 T_e = \mathrm{Tr}\big(D_s^2\big)$, denoted by X_0 for convenience in what follows, and differentiate it several times in sequence, the effect of pressure forces being neglected: this gives rise to a closed differential system of five equations for five unknowns:

$$\frac{\mathrm{d}X_0}{\mathrm{d}u} = 2X_1 \tag{5.5a}$$

$$\frac{\mathrm{d}X_1}{\mathrm{d}u} = X_2 - \tfrac{2}{3}TX_0 \tag{5.5b}$$

$$\frac{\mathrm{d}X_2}{\mathrm{d}u} = -\tfrac{4}{3}TX_1 \tag{5.5c}$$

$$\frac{\mathrm{d}T}{\mathrm{d}u} = 3P \tag{5.5d}$$

$$\frac{\mathrm{d}P}{\mathrm{d}u} = -\tfrac{2}{3}T^2 \tag{5.5e}$$

where the auxiliary variables X_1, X_2, T, P may be identified with

$$\begin{aligned} X_n &= \mathrm{Tr}\big(D_s v_s^n D_s\big) \qquad (n = 0, 1, 2) \\ T &= -\tfrac{1}{2}\,\mathrm{Tr}\big(v_s^2\big) \\ P &= \det(v_s). \end{aligned} \tag{5.6}$$

T and P are, in fact, *the characteristic coefficients* of v_s:

$$v_s^3 + Tv_s - P = 0. \tag{5.7}$$

The above system presents several integrals of the motion, among which the energy

$$2\hat{E} = X_0X_2 - X_1^2 \tag{5.8}$$

(which, in the present case, is of course purely kinetic energy). We also remark that the last two equations, equations (5.5d) and (5.5e), themselves constitute a closed sub-system, which admits the first integral (of the sixth degree in the momenta):

$$I_6^6 = 27P^2 + 4T^3 \tag{5.9}$$

where I_6^6 is precisely the discriminant of the characteristic equation (5.7). As it turns out, *it constitutes the highest degree term* (in the momenta) *of a new integral of the motion of the complete system*, including the effect of thermal pressure. It generalizes our earlier result (Gaffet 2000), which was obtained under the restricting assumption of rotation around a fixed principal axis, i.e. the case where F is block-diagonal.

To narrow down the search for the last missing integral, we observe that it should always vanish in block-diagonal cases, as those cases have not been found to be super-integrable. This leads us to consideration of the 3-vector:

$$\vec{A} = \vec{J} \wedge \tilde{V}_s\big[\vec{J}\big] \tag{5.10}$$

where $\tilde{V}_s\big[\vec{J}\big]$ is the 3-vector with components $\tilde{V}_{sik} J_k$: the vector $\vec{A}$ obviously vanishes in block-diagonal cases; in addition, it gives rise to an integral of free motion, as we now show. Differentiation of $\vec{A}$ (using the expression (5.3) of $\mathrm{d}\tilde{V}_s/\mathrm{d}u$) gives

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}u} = \vec{J} \wedge \frac{\mathrm{d}\tilde{V}_s}{\mathrm{d}u}\big[\vec{J}\big] = -\vec{J} \wedge \tilde{V}_s^2\big[\vec{J}\big] \tag{5.11}$$

and, by further differentiation

$$\frac{\mathrm{d}^2\vec{A}}{\mathrm{d}u^2} = 2\vec{J} \wedge \tilde{V}_s\big[\tilde{V}_s^2 - \tfrac{1}{3}\,\mathrm{Tr}\big(\tilde{V}_s^2\big)\big]\big[\vec{J}\big]. \tag{5.12}$$

As $\tilde{V}_s$ and v_s share the same characteristic equation (5.7), this shows that $\mathrm{d}^2\vec{A}/\mathrm{d}u^2$ is parallel to $\vec{A}$, so that $(\vec{A} \wedge \mathrm{d}\vec{A}/\mathrm{d}u)$ *is an integral of free motion, vanishing in block-diagonal cases*, and of degree seven in the momenta.

However, it is easy to show that $(\vec{A} \wedge \mathrm{d}\vec{A}/\mathrm{d}u)$ is parallel to $\vec{J}$:

$$\left(\vec{A} \wedge \frac{\mathrm{d}\vec{A}}{\mathrm{d}u}\right) = -\vec{J}\big(\vec{V}_0, \vec{V}_1, \vec{V}_2\big)$$

where $\vec{V}_n \equiv \tilde{V}_s^n\big[\vec{J}\big]$ ($n = 0, 1, 2$; note, in particular, that $\vec{V}_0 \equiv \vec{J}$). Thus the constancy of $\vec{A} \wedge \mathrm{d}\vec{A}/\mathrm{d}u$ merely reflects the fact that the triple product $\big(\vec{V}_0, \vec{V}_1, \vec{V}_2\big)$ is a constant.

6. The last two integrals

We have already noted that the integrals of motion of the complete system, including the pressure forces, must have an integral of free motion for the highest degree term. Two such integrals have been found in the preceding section: I_6^6 and the triple product $(\vec{V}_0, \vec{V}_1, \vec{V}_2)$. The essential point is that, from knowledge of these terms, the terms of lower degree may be deduced through an over-determined integration process that may be solved by quadratures. By this method we have been able to determine the fourth-degree term I_6^4, and the second degree one I_6^2, thus obtaining the exact integral whose highest-degree term is I_6^6 (there being no term of zero degree).

The existence of this new integral, together with our earlier results for the block-diagonal case, strongly suggests that the system under study is completely integrable; Liouville integrability requires one and only one additional integral, L say, vanishing in block-diagonal cases. There seems to be no other choice than $\big(\vec{V}_0, \vec{V}_1, \vec{V}_2\big)$ as the leading term of L, which is thus predicted to be of degree six in the momenta.

Note added in proof. There is in fact another possibility, owing to the presence of a partial symmetry which changes D_5 to its inverse, while preserving V_s. To the triple product $(\vec{V}_0, \vec{V}_1, \vec{V}_2)$ corresponds by symmetry another triple product, which turns out to be the leading term of the last integral L, as will be shown in a forthcoming publication.

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